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ON A PROPERTY OF RIEMANN ZETA FUNCTION

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Abstract. The Riemann zeta function and his famous conjecture regarding the property of this function were presented in his 1859 paper, which was concerned about the distribution of prime numbers. In this article, we will prove some properties of Riemann zeta function and based on those results we will formulate Theorem. An exposition is given, partly historical and partly mathematical, of the Riemann zeta function and the associated Riemann hypothesis. Relevance of these investigations to the theory of the distribution of prime numbers is discussed.

Keywords: Riemann zeta function, Chebyshev's conjecture and the Chebyshev functions, gamma function.

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Introduction

Georg Friedrich Bernhard Riemann (1826–1866) first enrolled at the University of Goettingen in 1846 to study towards a degree in Theology (his father Friedrich Bernhard Riemann was a Lutheran minister), but he began studying mathematics under Carl Friedrich Gauss, who recommended Riemann to switch to the mathematical field. With his father's approval, he transferred to the University of Berlin in 1847, and studied under Carl Jacobi (1804–1851), Peter Gustav Lejeune Dirichlet (1805–1859), Jacob Steiner (1796–1863) and Gotthold Eisenstein (1823–52) (see [1–4] and the references therein). The main person who influenced him at that time was Dirichlet. After a two years' stay in Berlin, Riemann returned to Goettingen and his Ph.D. thesis on theory of complex variables supervised by Gauss was submitted in 1851.

1 The Riemann zeta function and theorems

The Riemann zeta function is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the Riemann hypothesis) that remain unproved to this day. The Riemann zeta function is denoted $\zeta(s)$ and is plotted above (using two different scales) along the real axis. We make the correspondence in order to state the following definition:

1. $\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\cos(y \ln n)}{n^\alpha} + \mathrm{i} \sum_{n=1}^{\infty} \frac{\sin(y \ln n)}{n^\alpha}$, $\alpha > 1$, $s = \alpha + \mathrm{i}y$

$$\left(\sum_{n=1}^{\infty} \frac{\cos(y \ln n)}{n^\alpha}\right)^2 = \left(\sum_{n=1}^{\infty} \frac{\cos(y \ln n)}{n^{\alpha/2}} \cdot \frac{1}{n^{\alpha/2}}\right)^2 \le \sum_{n=1}^{\infty} \frac{\cos^2(y \ln n)}{n^\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \tag{1}$$

$$\left(\sum_{n=1}^{\infty} \frac{\sin(y \ln n)}{n^\alpha}\right)^2 \le \sum_{n=1}^{\infty} \frac{\sin^2(y \ln n)}{n^\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \tag{2}$$

Adding both sides of the inequality (1) and (2), we get

$$\left(\sum_{n=1}^{\infty} \frac{\cos(y \ln n)}{n^\alpha}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{\sin(y \ln n)}{n^\alpha}\right)^2$$
$$\le \left(\sum_{n=1}^{\infty} \frac{\cos(y \ln n)}{n^\alpha}\right)^2 - \left(\mathbf{i} \sum_{n=1}^{\infty} \frac{\sin(y \ln n)}{n^\alpha}\right)^2$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}} \le \left(\sum_{n=1}^{\infty} \frac{1}{n^\alpha}\right)^2.$$

Since the following inequality holds

$$\frac{1}{r} \cdot \frac{1}{(n+1)^r} \le \sum_{R=n+1}^{\infty} \frac{1}{R^{1+r}} \le \frac{1}{r} \cdot \frac{1}{n^r},\ \alpha = 1 + r,\ r > 0 \tag{3}$$

we have

$$0 \le \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}} \le \left(1 + \frac{1}{r}\right)^2$$

and hence

$$0 \le \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}}} \le 1 + \frac{1}{r}. \tag{4}$$

2. Let $\mu(n)$ be the Möbius function. Then $\mu(1) = 1$, and $\mu(n) = (-1)^k$ if n is the product of k prime numbers, and $\mu(n) = 0$ if n is divisible by the whole square of any prime number. Since $\frac{1}{\zeta(s)} = \sum\limits_{n=1}^{\infty} \frac{\mu(n)}{n^s}$,we get

$$\frac{1}{|\zeta(s)|} = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right|$$

and

$$|\zeta(s)| = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}}}}.$$

According to (4), we have

$$\frac{1}{|\zeta(s)|} \geq \frac{r}{r+1}.$$

Therefore, we have

$$\frac{r}{r+1} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{\mu(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\alpha}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \leq 1 + \frac{1}{r}$$

and hence

$$\frac{r}{r+1} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq 1 + \frac{1}{r}, \tag{5}$$

and

$$\frac{r}{r+1} \leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}}}} \leq 1 + \frac{1}{r}. \tag{6}$$

3. $\xi^2(s) = \sum\limits_{n=1}^{\infty} \frac{\tau(n)}{n^s}$.

Let $\tau(n)$ the number of divisors of n. Then

$$|\xi(s)| = \frac{1}{|\zeta(s)|} \cdot \left| \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \cdot \left| \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right|$$
$$= \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}}}}.$$

According to (6),

$$\frac{r}{r+1} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right| \leq 1 + \frac{1}{r},\ r > 0.$$

4. Let $r \geq 1$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we get

$$\frac{6}{\pi^2} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \frac{\pi^2}{6} \ \Rightarrow \ \frac{6}{\pi^2} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right| \leq \frac{\pi^2}{6}$$

and hence

$$\frac{6}{\pi^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}} \leq \frac{\pi^2}{6}.$$

For $r \geq 1$ as above done, we also get

$$\frac{r}{r+1} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq 1 + \frac{1}{r}$$

and

$$\frac{r}{r+1} \leq \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{\bar{s}}}} \leq 1 + \frac{1}{r},$$

where $s = \alpha + \mathbf{i}\beta,\ \alpha = r + 1,\ r > 0$. Hence we have

$$\frac{r}{r+1} \leq \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \right| \leq 1 + \frac{1}{r}.$$

Lemma 1. *The following equality holds*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = \left(1 - 2^{1-s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

in the convergence interval of the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^s}$. *Here* $s = \alpha - \boldsymbol{i}\beta$.

Proof. Since

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} + 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s},$$

we get

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}$$

and

$$\left(1 - 2^{1-s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}.$$

□

Lemma 2. *Let* $\alpha = 1$. *Then* $1 - 2^{1-s} = 0$.

Proof. Let

$$\begin{aligned} 1 - 2^{1-s} = 1 - 2 \cdot 2^{-s} &= 1 - 2 \cdot 2^{-\alpha - \mathbf{i}\beta} \\ &= 1 - 2^{1-\alpha} \cdot 2^{-\mathbf{i}\beta} \\ &= 1 - 2^{1-\alpha} (\cos \beta \ln 2 - \mathbf{i} \sin \beta \ln 2) = 0. \end{aligned}$$

It yields

$$\begin{cases} 1 - 2^{1-\alpha} \cos \beta \ln 2 = 0 \\ 2^{1-\alpha} \sin \beta \ln 2 = 0 \end{cases}.$$

Since $2^{1-\alpha} \neq 0$, we have

$$\sin \beta \ln 2 = 0 \Rightarrow \beta \ln 2 = \pi k, \ k = 0, \pm 1, \pm 2, \ldots$$

and $1 - 2^{1-\alpha} \cos \pi k = 0$. Hence $\cos \pi k = \dfrac{1}{2^{1-\alpha}}$. Since $\cos \pi k = \pm 1$ and $\frac{1}{2^{1-\alpha}} > 0$, we get $\cos \pi k = 1$. Therefore, $\dfrac{1}{2^{1-\alpha}} = 1$ and $\alpha = 1$. □

Corollary 3. *Let* $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n^s} \neq 0$ *and* $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^s} \neq 0$. *Since* $-2^{1-\alpha} \neq 0$, *we have* $1 - 2^{1-\alpha} \neq 1$. *It yields* $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n^s} \neq \pm \sum\limits_{n=1}^{\infty} \dfrac{1}{n^s}$.

Lemma 4. *Let* $0 < \alpha < 1$. *Then there is* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = 0$ *for all* $s \in \mathbb{C}$ *satisfying* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = 0$.

Proof. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} \neq 0$ for $\alpha \geq 1$. Assume that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = 0$ for some $0 < \alpha < 1$. $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{1}{n^s} = 0$ follows from Lemma 1. Then we get

$$\sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^s} = 0$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = 0.$$

For $\alpha = \frac{1}{2}$, there exits $y_0 \in \mathbb{R}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{\frac{1}{2} + \mathbf{i} y_0}} = 0$. □

It immediately follows $\sum_{n=1}^{\infty}\frac{1}{(2n)^s}=0$ since $\sum_{n=1}^{\infty}\frac{1}{(2n)^s}=\frac{1}{2^s}\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\frac{1}{2^s}\neq 0$.

Finally, we are summarizing the results above found in the following theorem.

Theorem 5. *If $\alpha\notin[0,\,1]$, then $\zeta(s)\neq 0$. (see [1] and the references therein)*

Corollary 6. *There exists a real number $\alpha_0=\Re\{s\}$ such that*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}=0$$

as long as $\sum_{n=1}^{\infty}\frac{1}{n^s}=0$. When $\alpha=1$, there exists a real number y_0 such that the series $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges for $s=1+\boldsymbol{i}y_0$ and

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}=0$$

.

We also have $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}=\left(1-2^{1-s}\right)\sum_{n=1}^{\infty}\frac{1}{n^s}$ in the convergence interval of $\sum_{n=1}^{\infty}\frac{1}{n^s}$. Hence there exists $s=1+\mathbf{i}\beta$, $\beta\in\mathbb{R}$ such that

$$\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}}{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}}=0.$$

Lemma 7. *Let $\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}, a_n\geq 0$ be the Dirichlet convergent series. Then*

$$\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}\cdot\sum_{n=1}^{\infty}a_ne^{-\lambda_n\overline{z}}\leq\left(\sum_{n=1}^{\infty}\frac{a_n}{e^{\lambda_n x}}\right)^2.$$

Proof. Let $z=x+\mathbf{i}y$, $x\geq 0$.

$$\sum_{n=1}^{\infty}a_ne^{-\lambda_n x-\mathbf{i}\lambda_n y}=\sum_{n=1}^{\infty}a_ne^{-\lambda_n x}\cdot e^{-\mathbf{i}\lambda_n y}=\sum_{n=1}^{\infty}a_ne^{-\lambda_n x}\cdot\left(\cos\left(\lambda_n y\right)-\mathbf{i}\sin\left(\lambda_n y\right)\right).$$

Since the series converges for $a_n \geq 0$, we have

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n \overline{z}} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \cos(\lambda_n y) + \mathbf{i} \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \sin(\lambda_n y)$$

and

$$\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \cos(\lambda_n y) \right)^2 \leq \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \cos^2(\lambda_n y) \cdot \sum_{n=1}^{\infty} \frac{a_n}{e^{\lambda_n x}}$$

and

$$\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \sin(\lambda_n y) \right)^2 \leq \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \sin^2(\lambda_n y) \cdot \sum_{n=1}^{\infty} \frac{a_n}{e^{\lambda_n x}}.$$

Adding both sides of the latter two inequality respectively, we get

$$\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \cos(\lambda_n y) \right)^2 + \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \cdot \sin(\lambda_n y) \right)^2$$
$$= \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \cdot \sum_{n=1}^{\infty} a_n e^{-\lambda_n \overline{z}} \leq \left(\sum_{n=1}^{\infty} \frac{a_n}{e^{\lambda_n x}} \right)^2.$$

□

Conclusions

We saw that the zeroes of the Riemann zeta function could be divided into the trivial ones and the non-trivial ones. The Riemann hypothesis says that they all lie one the line. This conjecture immediately would imply the Riemann hypothesis. One could construct such Hermitian operator by quantizing a Hamiltonian corresponding to a dynamical system.

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О СВОЙСТВЕ ДЗЕТА-ФУНКЦИИ РИМАНА

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Аннотация. Дзета-функция Римана и его знаменитая гипотеза относительно свойства этой функции были представлены в его статье 1859 г., в которой речь шла о распределении простых чисел. В этой статье мы докажем некоторые свойства дзета-функции Римана, и на основе этих результатов сформулируем теорему. Дано частично историческое и математическое описание дзета-функции Римана и связанной с ней гипотезы Римана. Обсуждается актуальность этих исследований для теории распределения простых чисел.

Ключевые слова: дзета-функция Римана, гипотеза Чебышева, функции Чебышева, гамма-функция.

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