Scientific article

UDC 51-7
DOI: 10.18101/2304-5728-2024-2-43-52

A Mathematical Model to Develop a Nomadic Livestock Connection With Industrial Objects

© **Ankhbayar Gelegbadam**
Ph.D (Math.), A/Professor,
Department of Applied Mathematics
National University of Mongolia
1 Ikh Surguuliin Gudamj, 14201, Ulaanbaatar, Mongolia
ankhbayar.g@num.edu.mn

© **Dultuya Terbish**
Ph.D (Math.), A/Professor,
Department of Applied Mathematics
National University of Mongolia
1 Ikh Surguuliin Gudamj, 14201, Ulaanbaatar, Mongolia
dultuya@num.edu.mn

© **Tserennadmid Tumurbaatar**
Ph.D (Computer Sci.), A/Professor,
Department of Information and Computer Sciences
National University of Mongolia
1 Ikh Surguuliin Gudamj, 14201, Ulaanbaatar, Mongolia
tserennadmid@num.edu.mn

Abstract. In this work, we have considered industrial objects and livestock enterprises, which are located in a given area. Some conditions and connections between them in the mathematical model are formulated newly, and the optimal equilibrium ratio states for the long-term existence of these objects are theoretically determined. Also, based on the mathematical models of the two objects, the optimal area ratio and values of the model were found.

Keywords: the mathematical model, nomadic livestock, industrial objects, optimal equilibrium ratio points, optimal control, optimal solution for the area.

For citation
Ankhbayar G., Dultuya T., Tserennadmid T. A Mathematical Model to Develop a Nomadic Livestock Connection With Industrial Objects // Bulletin of Buryat State University. Mathematics, Informatics. 2024. N. 2. P. 43–52.

Introduction

In Mongolia, nomadic livestock has been developed and practiced since ancient times. Currently, more than 90% of nomadic livestock is still traditional. Nowadays, the expansion of urban areas, the extraction and production of mining resources are intensifying in our country. This is having a negative impact on our traditional livestock farming. In particular, the development of nomadic husbandry in accordance with current needs and requirements is a priority.

Therefore, we will formulate mathematical models based on an ecological-economic model [1] and will be executed qualitative analysis. Due to intensive mining and other industries, the environment is polluted. In [1], the problem of keeping the pollution level at a certain level by devoting an amount of funds to environmental protection was studied. However, the level of pollution is a very general concept, and the interests of industrial enterprises that directly benefits from nature are always affected by conduct production on the territory where it exists. For this reason, in order to maintain the natural balance at a certain level, let's the amount of investment for natural restoration or protection the industrial object's is $K_2(t)$, and the ecological capacity of the livestock enterprise is $S(t)$. Also let's the production volume of the industrial object is $Y(t)$. Then the following balance equation can be written:

$$\dot{S}(t) = -m \cdot Y(t) + h \cdot K_2(t) \tag{1}$$

Here, m is the constant that indicate how intensively the production has a negative impact on nature and h is the constant that benefit of natural protection work.

Using (1) equation, we formulate the mathematical model of livestock enterprises and industrial objects, and the analysis of their optimal equilibrium ratio points and the research results of determining the optimal land ratio are presented in the following sections.

1 Model of nomadic livestock enterprises and industrial objects

1.1 Model of Industrial objects

The amount of $Y(t)$ is represented by the Cobb-Douglas production function:

$$Y(t) = A \cdot K_1^{\alpha}(t) \cdot L^{1-\alpha}(t), \quad (0 < \alpha < 1) \tag{2}$$

In (2), let the size of main fund of the industry be $K_1(t)$ and the amount of fund devoted to nature restoration be $K_2(t)$.

It is assumed that u_1 and u_2 of the produced products are used to increase the amount of funds for production and nature protection, respectively. Therefore, the mathematical model of the long-term profitable operation of the industrial object is formulated in following equation [1], [2], [4], :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(1 - u_1(t) - u_2(t)\Big) \cdot \frac{Y(t)}{L(t)} dt \to \max \tag{3}$$

$$\begin{aligned} \dot{K}_1(t) &= u_1(t)Y(t) - \mu K_1(t) \\ \dot{K}_2(t) &= u_2(t)Y(t) - \mu K_2(t) \\ \dot{L}(t) &= g \cdot L(t) \\ \dot{S}(t) &= -mY(t) + hK_2(t) \\ 0 \le u_1(t) \le 1,\ & 0 \le u_2(t) \le 1,\ u_1(t) + u_2(t) \le 1 \end{aligned} \tag{4}$$

Here, $g \ge 0$ is the growth coefficient of $L(t)$ workforce, μ is the loss coefficient of main fund, $S(t)$ is the grazing area assigned to the livestock enterprise. But $u_1(t)Y(t)$ and $u_2(t)Y(t)$ are the amount of investment for industry production and nature protection or restoration. Then

$$0 \le 1 - u_1(t) - u_2(t) \le 1$$

is the percentage of funds for consumption and

$$[1 - u_1(t) - u_2(t)] \cdot \frac{Y(t)}{L(t)}$$

is an indicator of average consumption per person.

In above model, the boundary conditions are not specifically written because we study the optimal versions of the equilibrium ratio that keeps $K_1(t), K_2(t), S(t)$ at a constant level. Due to we use the long-term profitable production represented by the functional (3).

In the model (3), (4):

$$K(t) = \frac{K_1(t)}{L(t)},\quad \overline{K}(t) = \frac{K_2(t)}{L(t)}$$

these conversions are done, it will be changed to the following form:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(1 - u_1(t) - u_2(t)\Big) \cdot AK^{\alpha} dt \to \max \tag{5}$$

$$\begin{aligned} \dot{K}(t) &= u_1(t)AK^\alpha - (\mu + g)K(t) \\ \dot{\overline{K}}(t) &= u_2AK^\alpha - (\mu + g)\overline{K}(t) \\ \dot{S}(t) &= \Big(-mAK^\alpha(t) + h\overline{K}(t)\Big)L(t) \\ 0 \le u_1(t) \le 1,\ 0 &\le u_2(t) \le 1,\ u_1(t) + u_2(t) \le 1 \end{aligned} \tag{6}$$

1.2 Livestock enterprise model

We let N be the number of animals, $S(t)$ be the size of grazing area, and $y(t)$ be the density of grass per unit area, then the model can be written by formula (7)-(8). Therefore, the model[2] [3] of long-term profitable operation of the livestock enterprise is:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T p(t)N dt \to \max \tag{7}$$

$$\begin{aligned} \dot{y}(t) &= g(y(t)) - F(y(t)) \cdot \frac{N}{S(t)} \\ \dot{p}(t) &= aF(y(t)) - bp^\gamma(t),\ (0 < \gamma < 1) \end{aligned} \tag{8}$$

Here, a is the metabolism coefficient, b is the energy loss factor, and γ is the basic exchange rate, particularly, $\gamma = \frac{3}{4}$ is derived based on statistical analysis in [5] [6]. $g(y)$ is the plant regeneration function satisfying the conditions that $g(0) = g(\overline{y}) = 0,\ g''(y) < 0$. Also $F(y)$ is a function representing the amount of grass eaten by one animal that meets the conditions

$$F(0) = 0,\ F'(y) > 0,\ F''(y) < 0,\ \lim_{y\to\infty} F(y) = F < \infty.$$

and $p(t)$ is the average weight of the animal. A connection of two models (5) – (6) and (7) – (8) is shown by the following scheme in Fig. 1.

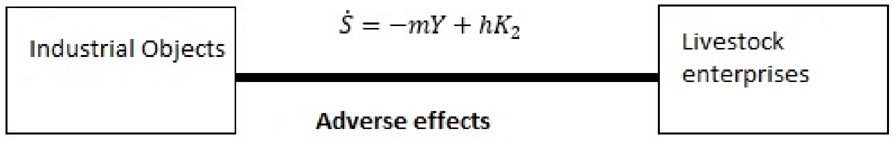

Fig. 1. Scheme of the model

2 Analysis of optimal equilibrium ratio points

Let $K(t) = \text{const},\ S(t) = \text{const}$ in (5) – (6), $y(t) = \text{const},\ p(t) = \text{const}$ in (7) – (8), the model turns to the following maximization problem:

$$(1 - u_1 - u_2)AK^\alpha \rightarrow\ max \tag{9}$$

$$\begin{aligned} u_1AK^\alpha - \eta K = 0,\ \eta = \mu + g \\ \overline{K} = \frac{m}{h}AK^\alpha \Rightarrow \overline{K}\cdot\left(\frac{u_2\cdot h}{m} - \eta\right) = 0 \\ 0 \le u_1 \le 1,\ 0 \le u_2 \le 1,\ u_1 + u_2 \le 1 \end{aligned} \tag{10}$$

$$\begin{aligned} p(t)\cdot N \rightarrow \max \\ g(y) - F(y)\cdot\frac{N}{S} = 0 \\ aF(y) - bp^\gamma = 0 \end{aligned} \tag{11}$$

From equation (10), $u_2 = \dfrac{\eta\cdot m}{h}$ is obtained. So $\dfrac{\eta\cdot m}{h} \le 1$.

a) If $\dfrac{\eta\cdot m}{h} = 1$, it will be impossible to carry out production as $\overline{K} = K_1 = 0$, all capital is devoted to nature protection or restoration.

b) Consider the case $u_1 + u_2 \le 1,\ u_2 = \dfrac{\eta\cdot m}{h} < 1$. In this case, production can be profitable. Also

$$u_1 = \frac{\eta}{A}\cdot K^{1-\alpha}, \quad \overline{K} = \frac{m}{h}AK^\alpha$$

and the objective function is formulated as follows

$$(1 - \eta_1 - \eta_2)A\left(\frac{u_1A}{\eta}\right)^{\frac{\alpha}{1-\alpha}} = \left(1 - u_1 - \frac{\eta\cdot m}{h}\right)A^{\frac{1}{1-\alpha}}\cdot\eta^{\frac{-\alpha}{1-\alpha}}\cdot u_1^{\frac{\alpha}{1-\alpha}}.$$

Therefore, we have to consider the following maximization problem:

$$\begin{aligned} M(u_1) = \left(1 - u_1 - \frac{\eta\cdot m}{h}\right)\cdot u_1^{\frac{\alpha}{1-\alpha}}\cdot A^{\frac{1}{1-\alpha}}\cdot\eta^{-\frac{\alpha}{1-\alpha}} \rightarrow \max \\ 0 \le u_1 \le 1 - \frac{\eta\cdot m}{h} \end{aligned}$$

An optimal solution exists of above problem and it is found:

$$M'(u_1) = 0 \Rightarrow\ -u_1^{\frac{\alpha}{1-\alpha}} + (1 - u_1 - \frac{\eta\cdot m}{h})\cdot\frac{\alpha}{1-\alpha}\cdot u_1^{\frac{2\alpha-1}{1-\alpha}} = 0\ \Rightarrow u_1 = \alpha\left(1 - \frac{\eta m}{h}\right).$$

And from equation (11),

$$p(t) = \left(\frac{a}{b}\right)^{\frac{1}{\gamma}} \cdot F^{\frac{1}{\gamma}}(y), \quad N = S \cdot \frac{g(y)}{F(y)}$$

is found and the objective function is formulated by

$$Np(t) = S \cdot \left(\frac{a}{b}\right)^{\frac{1}{\gamma}} \cdot F^{\frac{1-\gamma}{\gamma}}(y) \cdot g(y).$$

Therefore, it is necessary to consider the following maximization problem:

$$L(y) = \left(\frac{a}{b}\right)^{\frac{1}{\gamma}} \cdot F^{\frac{1-\gamma}{\gamma}}(y) \cdot g(y) \to \max$$

The maximum point of the function $L(y)$ is to the right of maximum point $g(y)$ function's. Because

$$\frac{1-\gamma}{\gamma} > 0 \Rightarrow g'(\overline{y}) = 0 \Rightarrow \overline{y} < y_{op} \in L(y).$$

and $g'(y_{op}) < 0$. Now let's study the stability of system equations (11) for stationary points (y_{op}, p_{op}). For this, we write the Jacobian matrix:

$$\begin{pmatrix} g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} & 0 \\ a \cdot F'(y_{op}) & -b\gamma p_{op}^{\gamma-1} \end{pmatrix}$$

and the characteristic equation is

$$\lambda^2 - (g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} - b\gamma p_{op}^{\gamma-1})\lambda - (g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S})b\gamma p_{op}^{\gamma-1} = 0$$

and its solution is

$$\lambda_{1,2} = \frac{g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} - b\gamma p_{op}^{\gamma-1} \pm \sqrt{D}}{2}$$
$$= \frac{g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} - b\gamma p_{op}^{\gamma-1} \pm \left[g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} + b\gamma p_{op}^{\gamma-1}\right]}{2}$$

and

$$\lambda_1 = g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} < 0, \quad \lambda_2 = -b\gamma p_{op}^{\gamma-1} < 0$$

here,

$$D = \left[g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S} - b\gamma p_{op}^{\gamma-1}\right]^2 + 4\left(g'(y_{op}) - F'(y_{op}) \cdot \frac{N}{S}\right)b\gamma p_{op}^{\gamma-1}.$$

Therefore, the system is stable and (y_{op}, p_{op}) becomes the stabilization point. We summarize the above results and write the following corollary.

Corollary 2.1. *Long-term steady-state points for both two models in (5)-(8) are*

$$K_1 = \left(\frac{u_1 A}{\eta}\right)^{\frac{1}{1-\alpha}} \cdot L(0)e^{gt}, \quad K_2 = K_2(0) \cdot e^{\left(\frac{u_2 h}{m} - \mu\right)t}, \quad (y_{op},\ p_{op})$$

and optimal control is

$$u^* = u_1 + u_2 = \frac{\eta m}{h}(1-\alpha) + \alpha.$$

Also, the optimal adjustment of the unit area is made by the equation:

$$\frac{N_{op}}{S} = \frac{g(y_{op})}{F(y_{op})}.$$

3 The case where the industrial object requires land

The previous sections included the negative impact that industry objects and the amount of products it produces is reduce the grazing area of livestock enterprises. But we did not include in detail the case of demand another grazing area. To consider this case, suppose that the amount of products it produces depends on three main factors: the grazing area S_1, the main fund for production K_1, and the labor force or population L_1. Then

$$Y(S_1, K_1, L_1) = AS_1^{\beta} K_1^{\alpha} L_1^{1-\alpha}, 0 \leq \alpha, \beta \leq 1.$$

It provides the definition of the production function. Let be the number of people working at the livestock enterprise is L_2 and be the total area between the industrial object and the livestock enterprise is S_{max}. Then $S = S_{max} - S_1$. Here, S is the area per livestock enterprises.

If we consider both the industrial object and the livestock enterprise as a monolithic system operating on its own area and set a long-term profitable operation goal, we can write the following objective function. Where

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T (1 - u_1(t) - u_2(t)) \cdot \frac{Y(t)}{L_1(t)}dt + \int_0^T \frac{p(t)N}{L_2(t)}dt\right) \to \max \qquad (12)$$

If we initially assume that $L_1(t) = \text{const}$, $L_2(t) = \text{const}$, $S(t) = \text{const}$ and $S_1(t) = \text{const}$, each increment of functional (12) take to the maximum value from the research in section 2 when $\frac{\eta m}{h} < 1$. That maximum value is obtained as follows

$$u_1^{op} = \alpha(1 - \frac{\eta m}{h}), \ \ u_2^{op} = \frac{\eta m}{h}.$$

and optimal values of grass density y^{op} are found. In other words, for the constants S and S_1, whose sum is S_{max}, each increment of the functional (12) takes its maximum value in the following case:

$$\left(1-u_1^{op}-u_2^{op}\right)\cdot A^{\frac{1}{1-\alpha}}\cdot\eta^{\frac{-\alpha}{1-\alpha}}\cdot\left(u_1^{op}\right)^{\frac{2\alpha-1}{1-\alpha}}\cdot S_1^{\frac{\beta}{1-\alpha}}+$$
$$+(S_{max}-S_1)\cdot\left(\frac{a}{b}\right)^{\frac{1}{\gamma}}\cdot\frac{g(y^{op})}{L_2}\cdot\left(F(y^{on})\right)^{\frac{1}{\gamma}-1}.$$

If we use the notation as

$$\left(1-u_1^{op}-u_2^{op}\right)\cdot A^{\frac{1}{1-\alpha}}\cdot\eta^{\frac{-\alpha}{1-\alpha}}\cdot\left(u_1^{op}\right)^{\frac{2\alpha-1}{1-\alpha}}=C_1>0,$$

$$\left(\frac{a}{b}\right)^{\frac{1}{\gamma}}\cdot\frac{g(y^{op})}{L_2}\cdot\left(F(y^{op})\right)^{\frac{1}{\gamma}-1}=C_2>0,$$

following optimization problem can be set. It includes:

$$M(S_1)=C_1\cdot S_1^{\frac{\beta}{1-\alpha}}+(S_{max}-S_1)\cdot C_2\to\max$$
$$0\le S_1\le S_{max}$$

In here if $\frac{\beta}{1-\alpha}\geq 1$ or $\frac{\beta}{1-\alpha}<1$, either the industrial object or the livestock enterprise is not needed when

$$C_1\cdot(S_1^{op})^{\frac{\beta}{1-\alpha}}+(S_{max}-S_1^{op})\cdot C_2<\max(S_{max}\cdot C_2, C_1\cdot S_{max}^{\frac{\beta}{1-\alpha}}).$$

But when $\alpha+\beta<1$ and

$$C_1\cdot(S_1^{op})^{\frac{\beta}{1-\alpha}}+(S_{max}-S_1^{op})\cdot C_2\geq\max(S_{max}\cdot C_2, C_1\cdot S_{max}^{\frac{\beta}{1-\alpha}})$$

the conditions are met, the optimal solution for the area is obtained as follows

$$S_1^{op}=\left(\frac{1-\alpha}{\beta}\cdot\frac{C_2}{C_1}\right)^{-\frac{1-\alpha}{1-\alpha-\beta}}.$$

Conclusion

In the first section of this work, we formulated of the mathematical models of the production object and the livestock enterprise located in a limited area, and the balance equation (1) expressing the connection between these two objects was newly formulated. In this regard, some conditions of the mathematical model were formulated. Also, we analyzed the points of an optimal equilibrium ratio of the models (5)-(6),(7)-(8), and the stable state points for the long-term existence of the two objects were found. In this regard, we found the optimal control of the model. As well as, we formulated the corollary. In section 3, the mathematical formulation of the proper amount of the area in two objects is presented.

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The article was submitted 27.04.2024; approved after reviewing 20.06.2024; accepted for publication 23.09.2024.

МАТЕМАТИЧЕСКАЯ МОДЕЛЬ РАЗВИТИЯ СВЯЗИ КОЧЕВОГО ЖИВОТНОВОДСТВА С ПРОМЫШЛЕННЫМИ ОБЪЕКТАМИ

Г. Анхбаяр
Ph. D., профессор,
кафедра прикладной математики
Национальный университет Монголии
Монголия, 14201, г. Улан-Батор, Их Сургуулийн гудамж, 1
ankhbayar.g@num.edu.mn

Т. Дултуя
Ph. D., профессор,
кафедра прикладной математики
Национальный университет Монголии
Монголия, 14201, г. Улан-Батор, Их Сургуулийн гудамж, 1
dultuya@num.edu.mn

Т. Цереннадмид
Ph. D., профессор,
кафедра информатики и компьютерных наук
Национальный университет Монголии
Монголия, 14201, г. Улан-Батор, Их Сургуулийн гудамж, 1
tserennadmid@num.edu.mn

Аннотация. В данной работе рассмотрены промышленные объекты и животноводческие предприятия, расположенные на заданной территории. Сформулированы заново некоторые условия и связи между ними в математической модели, а также теоретически определены оптимальные соотношения равновесных состояний для длительного существования этих объектов. Также на основе математических моделей двух объектов найдены оптимальные соотношения площадей и параметры модели.

Ключевые слова: математическая модель, кочевой скот, промышленные объекты, точки оптимального соотношения равновесия, оптимальное управление, оптимальное решение для области.

Для цитирования
Анхбаяр Г., Дултуя Т., Цереннадмид Т. Математическая модель развития связи кочевого животноводства с промышленными объектами // Вестник Бурятского государственного университета. Математика, информатика. 2024. № 2. С. 43–52.

Статья поступила в редакцию 27.04.2024; одобрена после рецензирования 20.06.2024; принята к публикации 23.09.2024.